Modeling Dividends and Other Distributions
The Effect Of Dividends On Future Asset Price

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Company capital that is over and above what is needed to finance the company's current operations is paid out over time as dividends and/or other distributions. This excess capital can be distributed to shareholders, retained by the company, or both. For our purposes we will define dividends to be the distribution of excess capital whether that excess capital comes from earnings or from some other source (non-earnings related distributions).

In this white paper we model the cash account used to accumulate and reinvest dividends and other distributions retained by the company. To that end we will use the following hypothetical problem...

Our Hypothetical Problem

We are tasked with building a model to model the dividends and other distributions cash account. We are given the following go-forward model assumptions...

Table 1: Go-Forward Model Assumptions

Symbol	Description	Value
C_0	Annualized net cash flow at time zero ($)	1,000,000
λ	Expected cash flow growth rate - mean (%)	4.00
σ	Expected cash flow growth rate - volatility (%)	18.00
κ	Risk-adjusted discount rate (%)	12.00

Assume that the rates in the table above are continuous-time rates. Our task is to answer the following questions:

Question 1: What is the value of the company at time zero?

Question 2: What is the expected value of the company at the end of year 3?

Modeling Cash Flow Over Time

We will define the variable C_t to be annualized cash flow at time t and the variable δW_t to be the change in the underlying brownian motion over time. Using the parameters in Table 1 above the stochastic differential equation for the change in annualized cash flow over the time interval $[t, t + \delta t]$ is...

$$\delta C_t = \lambda \, C_t \, \delta t + \sigma \, C_t \, \delta W_t \ \text{ ...where... } \ \delta W_t \sim N\left[0, \delta t\right] \tag{1}$$

The solution to Equation (1) above is the equation for random annualized cash flow at time t, which is...

$$C_t = C_0 \, \text{Exp}\left\{ \left(\lambda - \frac{1}{2}\,\sigma^2 \right) t + \sigma \sqrt{t} \, Z \right\} \ \text{ ...where... } \ Z \sim N\left[0, 1\right] \tag{2}$$

Note that annualized cash flow in Equation (2) above is a lognormally-distributed random variable and as such has the following expectation...

$$\mathbb{E}\left[C_t\right] = C_0 \, \text{Exp}\left\{ \text{Mean} + \frac{1}{2}\,\text{Variance} \right\} = C_0 \, \text{Exp}\left\{ \left(\lambda - \frac{1}{2}\,\sigma^2 \right) t + \frac{1}{2}\,\sigma^2 t \right\} = C_0 \, \text{Exp}\left\{ \lambda \, t \right\} \tag{3}$$

Modeling Asset Value Over Time

We will define the variable V_t to be asset value at time t. Using Equation (3) above and the parameters in Table 1 above the equation for asset value is...

$$V_t = \int_t^{\infty} \mathbb{E}\Big[C_u\Big] \operatorname{Exp}\Big\{ -\kappa\,(u-t)\Big\}\,\delta u = C_0 \operatorname{Exp}\Big\{\kappa\,t\Big\} \int_t^{\infty} \operatorname{Exp}\Big\{(\lambda-\kappa)\,u\Big\}\,\delta u \tag{4}$$

The solution to the integral in Equation (4) above is...

$$\int_t^{\infty} \operatorname{Exp}\Big\{(\lambda-\kappa)\,u\Big\}\,\delta u = \Big(\lambda-\kappa\Big)^{-1}\Big(\operatorname{Exp}\Big\{(\lambda-\kappa)\,\infty\Big\} - \operatorname{Exp}\Big\{(\lambda-\kappa)\,t\Big\}\Big) \tag{5}$$

Note that when $\lambda < \kappa$ we have the following limit...

$$\lim_{t\to\infty} \operatorname{Exp}\Big\{(\lambda-\kappa)\,t\Big\} = 0 \ \text{ ...when... } \ \lambda < \kappa \tag{6}$$

Using Equation (6) above we can rewrite Equation (5) above as...

$$\int_t^{\infty} \operatorname{Exp}\Big\{(\lambda-\kappa)\,u\Big\}\,\delta u = \Big(\lambda-\kappa\Big)^{-1}\Big(0 - \operatorname{Exp}\Big\{\lambda\,t\Big\}\operatorname{Exp}\Big\{-\kappa\,t\Big\}\Big) = \Big(\kappa-\lambda\Big)^{-1}\operatorname{Exp}\Big\{\lambda\,t\Big\}\operatorname{Exp}\Big\{-\kappa\,t\Big\} \tag{7}$$

Using Equation (7) above we can rewrite valuation Equation (4) above as...

$$V_t = C_0 \operatorname{Exp}\Big\{\kappa\,t\Big\}\Big(\kappa-\lambda\Big)^{-1}\operatorname{Exp}\Big\{\lambda\,t\Big\}\operatorname{Exp}\Big\{-\kappa\,t\Big\} = C_0 \operatorname{Exp}\Big\{\lambda\,t\Big\}\Big(\kappa-\lambda\Big)^{-1} \tag{8}$$

The equation for the derivative of Equation (8) above with respect to the variable λ is...

$$\delta V_t = \lambda\,C_0 \operatorname{Exp}\Big\{\lambda\,t\Big\}\Big(\kappa-\lambda\Big)^{-1}\delta t = \lambda\,V_t\,\delta t \tag{9}$$

Note that Equation (9) says that expected asset price should increase at a rate of λ, which is the expected cash flow growth rate. We will define the variable ϕ to be the dividend rate. The expected annual return on the asset is therefore capital gains (λ) plus dividends (ϕ).

Deriving The Dividend Rate

We will define the variable μ to be the asset's expected total return and the variable ϕ as the asset's dividend yield. The stochastic differential equation that defines the change in asset price over time is...

$$\delta V_t = \mu\,C_t\,\delta t - \phi\,C_t\,\delta t + \sigma\,C_t\,\delta W_t \ \text{ ...where... } \ \delta W_t \sim N\Big[0, \delta t\Big] \tag{10}$$

The solution to Equation (10) above is the equation for random asset value at time t, which is...

$$V_t = V_0 \operatorname{Exp}\Big\{\Big(\mu - \phi - \frac{1}{2}\,\sigma^2\Big)\,t + \sigma\,\sqrt{t}\,Z\Big\} \ \text{ ...where... } \ Z \sim N\Big[0, 1\Big] \tag{11}$$

Note that annualized cash flow in Equation (11) above is a lognormally-distributed random variable and as such has the following expectation...

$$\mathbb{E}\Big[V_t\Big] = V_0 \operatorname{Exp}\Big\{\text{Mean} + \frac{1}{2}\,\text{Variance}\Big\} = V_0 \operatorname{Exp}\Big\{\Big(\mu-\phi-\frac{1}{2}\,\sigma^2\Big)t + \frac{1}{2}\,\sigma^2 t\Big\} = V_0 \operatorname{Exp}\Big\{(\mu-\phi)\,t\Big\} \tag{12}$$

Using Equations (8) and (12) above we have the following two equations for asset price...

$$V_t = C_0 \operatorname{Exp}\Big\{\lambda\,t\Big\}\Big(\kappa-\lambda\Big)^{-1} \ \text{ ...and... } \ V_t = V_0 \operatorname{Exp}\Big\{(\mu-\phi)\,t\Big\} \tag{13}$$

Using Equation (8) above asset price at time zero is...

$$V_0 = C_0 \,\mathrm{Exp}\left\{\lambda \times 0\right\}\left(\kappa - \lambda\right)^{-1} = C_0\left(\kappa - \lambda\right)^{-1} \tag{14}$$

Using Equation (14) above we can rewrite the equations in Equation (13) above as...

$$V_t = C_0 \,\mathrm{Exp}\left\{\lambda\, t\right\}\left(\kappa - \lambda\right)^{-1} \text{ ...and... } V_t = C_0\left(\kappa - \lambda\right)^{-1}\mathrm{Exp}\left\{(\mu - \phi)\, t\right\} \tag{15}$$

If we equate the equations in Equation (15) above we can derive the equation for the dividend yield (ϕ) as follows...

$$\begin{aligned} C_0 \,\mathrm{Exp}\left\{\lambda\, t\right\}\left(\kappa - \lambda\right)^{-1} &= C_0\left(\kappa - \lambda\right)^{-1}\mathrm{Exp}\left\{(\mu - \phi)\, t\right\} \\ \mathrm{Exp}\left\{\lambda\, t\right\} &= \mathrm{Exp}\left\{(\mu - \phi)\, t\right\} \\ \lambda\, t &= (\mu - \phi)\, t \\ \phi &= \mu - \lambda \end{aligned} \tag{16}$$

The Answers To Our Hypothetical Problem

Question 1: What is the value of the company at time zero?

Using the parameters in Table 1 above we can define the variables C_0, λ and κ as...

$$C_0 = 1,000,000 \text{ ...and... } \lambda = 0.04 \text{ ...and... } \kappa = 0.12 \tag{17}$$

Using the parameters in Equation (8) and (17) above the answer to the question is...

$$V_0 = C_0 \,\mathrm{Exp}\left\{\lambda\, t\right\}\left(\kappa - \lambda\right)^{-1} = 1,000,000 \times \mathrm{Exp}\left\{0.04 \times 0\right\} \times \left(0.12 - 0.04\right)^{-1} = 12,500,000 \tag{18}$$

Question 2: What is the expected value of the company at the end of year 3?

If we discount at a rate of κ then that rate is also the expected rate of future total return. In other words if we discount at a rate of κ to derive asset value today then that asset will earn a rate of return equal to κ in the future. Using the parameters in Table 1 above we can define the variable μ as...

$$\mu = \kappa = 0.12 \tag{19}$$

Using Equations (16) and (22) above and the parameters in Table 1 above we can define the variable ϕ as...

$$\phi = 0.12 - 0.04 = 0.08 \tag{20}$$

Using Equations (13), (18), (22) and (20) above the answer to the question is...

$$\mathbb{E}\left[V_3\right] = V_0 \,\mathrm{Exp}\left\{(\mu - \phi)\, t\right\} = 12,500,000 \times \mathrm{Exp}\left\{(0.12 - 0.08) \times 3\right\} = 14,094,000 \tag{21}$$

We can also use first part of Equation (13) above to get the same answer as in Equation (21) above...

$$\mathbb{E}\left[V_3\right] = C_0 \,\mathrm{Exp}\left\{\lambda\, t\right\}\left(\kappa - \lambda\right)^{-1} = 1,000,000 \times \mathrm{Exp}\left\{0.04 \times 3\right\}\left(0.12 - 0.04\right)^{-1} = 14,094,000 \tag{22}$$